Weak KAM methods in PDE

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In the last decade Weak KAM methods and tools have proved to be effective even when applied to Hamilton–Jacobi equations with Hamiltonians of so low regularity that an Hamiltonian flow cannot be defined.

Weak KAM Theory without Hamiltonian flow, and sometimes even without Hamiltonians, has found applications in different domains, often very afar from its original framework. The lists of subjects include large time behavior of time–dependent HJ equations, inverse Lyapunov theorems, construction of smooth time functions on Lorentzian manifolds, weakly coupled control systems, optimal transport, homogenization, singularly perturbed control systems, Mean field games theory, Wasserstein spaces.

The aim of the course is to give an account at least of some aspects of this domain of research. Roughly, it will be divided in three segments. The first will be of introductory character starting from the very definition of weak solutions in the two versions of Crandall–Lions and Fathi. The second part will cover the fundamentals of Weak KAM theory applied to continuous Hamilton–Jacobi equations, and the final part will be devoted to specific applications and models.

A more detailed outline of the topics we plan to cover is as follows:

- Weak solutions, viscosity test functions, calibrated curves, Lax–Oleinik semigroup.
- Eikonal and discounted HJ equations, intrinsic metric, representation formulas. Comparison and stability results.
- Critical equations, Weak KAM solutions, compact and noncompact ambient spaces.
- Aubry and Mather set, PDE and metric characterization. Mather measures. Regularity issues for subsolutions.
- Homogenization of HJ equation. Coercive and noncoercive case.
- Weakly coupled systems of HJ equations, critical value. Definition and properties of adapted Aubry set.
- Weak KAM theory in Wasserstein spaces

It goes without saying that this is just a tentative programm, that will be modified and adapted according to the actual pace of the lectures and to requests of the audience. In principle the course does not require any specific background in PDE, however some basic knowledge of Viscosity Solutions Theory as well as Weak KAM theory will help following the thread of the presentation.